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The local sine–Gordon model

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Abstract. Effects of spatial inhomogeneity in the one-dimensional quantum sine–Gordon model are treated by use of the variational Gaussian wave-functional method. The properties of the ground state and the phase diagram of the system are investigated in detail.

The one-dimensional (1D) quantum sine–Gordon system has been widely used to investigate the nonlinear excitation in condensed matter physics. Since impurities or defects are usually present in the material samples, the 1D space will no longer be homogeneous, this effect should be considered in the problem of nonlinear excitations introducing a local sine–Gordon model. The physical sense of this model emerges in its connections with the Kondo problem [1] and the spin-boson problem with ohmic dissipation [2] after the well known bosonization procedure of the fermi operators has been performed. In this paper we approach this problem with the Gaussian wave-functional technique which has been applied successfully in quantum field theory [3–5]. With this technique we study the properties of the ground states and derive the phase diagram of the system.

The Hamiltonian of the local sine–Gordon model is

$$H = \int \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x \phi)^2 - \frac{v\delta(x)}{\beta^2}(\cos\beta\phi - 1) \right\} \mathrm{d}x \tag{1}$$

where $\phi(x)$ and $\pi(x)$ are the bose field and its canonical conjugate momentum, respectively, and the potential term with $\delta(x)$ describes the spatial inhomogeneities due to the impurities [6].

In solving the eigenequation of the Hamiltonian (1), we appeal to the variational approach with a trial Gaussian wave functional

$$\begin{aligned} \Psi(\phi, \Phi, p, f) = N_f \exp \Bigg\{ \mathrm{i} \int P(x)\phi(x)\,\mathrm{d}x \\ -\frac{1}{2}\int [\phi(x) - \Phi(x)) f(x, y)(\phi(y) - \Phi(y)]\,\mathrm{d}x\,\mathrm{d}y \Bigg\} \end{aligned} \tag{2}$$

where $\Phi(x)$, $P(x)$, and $f(x, y)$ are variational parameters. The expectation value of the Hamiltonian (1) with respect to the wavefunction (2) has been given in [3–5]

$$\begin{aligned} E = \int \left\{ \frac{1}{2}P^2 + \frac{1}{2}(\partial_x\Phi)^2 - \frac{v\delta(x)}{\beta^2}(Z\cos\beta\Phi - 1) + \frac{1}{4}f(x, x) \right\} \mathrm{d}x \\ + \frac{1}{4}\int \delta(x - y)\frac{\partial^2 f^{-1}(x, y)}{\partial x \partial y}\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{3}$$

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where

$$Z = \exp\{-\beta^2 f^{-1}(x,x)/4\} \tag{4}$$

and $f^{-1}(x,y)$ denotes the inverse of $f(x,y)$. In the case of the local sine–Gordon model, the eigenstates are to be variationally imitated by those of a local quadratic Hamiltonian model, the variational functions take the form [7]

$$f(x,y) = \sum_{k_l>0} k_l u_l(x) u_l(y) \tag{5}$$

$$f^{-1}(x,y) = \sum_{k_l>0} \frac{1}{k_l} u_l(x) u_l(y) \tag{6}$$

where $u_l(x)$ is a set of complete orthogonal functions from the eigenequation with the boundary condition, $u_l(\pm\frac{1}{2}L) = 0$,

$$\mathcal{G} u_l = \left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2m\delta(x)\right] u_l(x) = k_l^2 u_l(x). \tag{7}$$

There are two solutions for a given k_l, explicitly

$$u_l(x) = \sqrt{\frac{2}{L}} \sin k_l x \qquad \text{or} \qquad \sqrt{\frac{2}{L-(1/k_l)\sin 2\delta_l}} \cos(k_l|x| + \delta_l) \tag{8}$$

where $\delta_l = -\tan^{-1}(m/k_l)$, m being the variational parameter.

Obviously, the state with vanishing momentum and constant $\Phi(x) = \Phi(0)$ has the lowest energy:

$$E = \frac{1}{2}\int f(x,x)\,\mathrm{d}x - \frac{v}{\beta^2}[Z(m)\cos\beta\Phi(0) - 1] - \frac{m}{2} f^{-1}(0,0) \tag{9}$$

where Z is replaced by its value at the origin, which depends on m via $f^{-1}(0,0)$

$$Z(m) = \exp\{-\beta^2 f^{-1}(0,0)/4\} \tag{10}$$

and in the continuum limit

$$f^{-1}(0,0) = \frac{2}{L}\sum_{k_l>0} \frac{k_l}{k_l^2 + m^2 + 2m/L} = \frac{1}{\pi}\int_0^{\Lambda} \frac{k}{k^2+m^2}\,\mathrm{d}k = \frac{1}{2\pi}\ln\frac{m^2+\Lambda^2}{m^2}. \tag{11}$$

Λ is the ultraviolet cut-off of the momentum. Minimizing the energy of equation (9) with respect to m gives m as a function of $\Phi(0)$ by the equation

$$m = \tfrac{1}{2} v Z(m) \cos\beta\Phi(0) \tag{12}$$

where use was made of the relation (Hellmann–Feynman theorem)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}m}\int f(x,x)\,\mathrm{d}x &= \sum_{k_l>0}\frac{1}{2k_l}\frac{\mathrm{d}}{\mathrm{d}m}\langle u_l|\mathcal{G}|u_l\rangle \\ &= \sum_{k_l>0}\frac{1}{2k_l}\left\langle u_l\left|\frac{\mathrm{d}\mathcal{G}}{\mathrm{d}m}\right|u_l\right\rangle = f^{-1}(0,0).\end{aligned} \tag{13}$$

Correspondingly, the effective potential is defined as

$$\mathcal{V}(\Phi(0)) = E(\Phi(0), m(\Phi(0))). \tag{14}$$

The minimum condition of vanishing derivative with respect to $\Phi(0)$ is

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\Phi(0)} = \frac{2m}{\beta}\tan\beta\Phi(0) = 0 \tag{15}$$

then $\Phi(0) = (2\pi/\beta)n$ with n integers, and it implies that the vacuum of the quantum local sine–Gordon model is degenerate; however, one can prove there is no tunnelling effect between these vacuum states and, as a result, we specialize the vacuum sector $\Phi(0) = 0$, and define the renormalized mass μ as

$$\mu = \frac{1}{2}\frac{\mathrm{d}^2\mathcal{V}}{\mathrm{d}^2\Phi(0)}\bigg|_{\Phi(0)=0} = \frac{1}{2}vZ(\mu). \tag{16}$$

The ground state is stable only in the case of $E(\mu)$ being a local minimum, so the stability condition is constrained by

$$\frac{\mathrm{d}^2E}{\mathrm{d}^2\mu} = \frac{\Lambda^2}{2\pi(\mu^2+\Lambda^2)\mu}\left[1 - \frac{\beta^2}{4\pi}\frac{\Lambda^2}{\mu^2+\Lambda^2}\right] \geqslant 0 \tag{17}$$

and the critical condition then follows as

$$1 - \frac{\beta^2}{4\pi}\frac{1}{1+\mu^2\Lambda^{-2}} = 0. \tag{18}$$

By eliminating $\mu\Lambda^{-1}$ in equations (16) and (18), one can obtain the boundary of phase separation in the parameter plane $(v\Lambda^{-1}, \gamma = \beta^2/4\pi)$ as

$$v\Lambda^{-1} = \frac{2\gamma^{\gamma/2}}{(\gamma-1)^{(\gamma-1)/2}}. \tag{19}$$

The boundary consists of a vertical line $(0 \leqslant v\Lambda^{-1} \leqslant 2, \gamma = 1)$ and a monotonous critical line of equation (19) which behaves as $2\mathrm{e}^{1/2}\gamma^{1/2}$ when $\gamma \sim \infty$. At the vertical line, the renormalized mass μ vanishes from the left, announcing a continuous phase transition. The critical line of equation (19) is connected with the first-order phase transition, and $v\Lambda^{-1} = 2$, $\gamma = 1$ refers to the tricritical point.

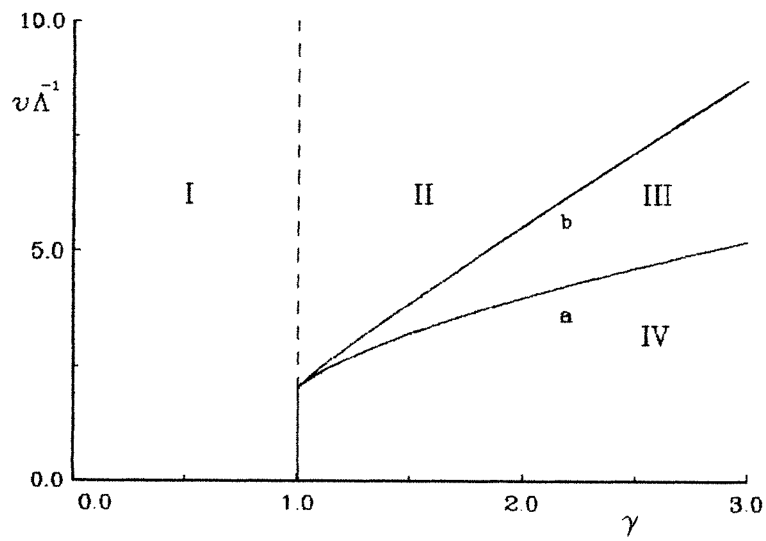

Figure 1. Phase diagram of the local sine–Gordon model. a is the critical line and b the equilibrium line.

In region I ($\gamma < 1$) of figure 1, there exist only the states with $\mu > 0$ at which $E(\mu)$ reaches its absolute minimum. Between the line of $\gamma = 1$ and the critical line, the energy of the state $E(\mu)$ reaches a local minimum, which may be higher or lower than that of the state $E(0)$ with $\mu = 0$. For determining the absolute minimum, we further divide the phase diagram into subdivisions by the equilibrium condition

$$E(\mu) = E(0) \tag{20}$$

namely,

$$\gamma = \frac{\tan^{-1} \mu\Lambda^{-1}}{\mu\Lambda^{-1}}. \tag{21}$$

Together with equation (16), the equilibrium line can be determined in the parameter plane. Across this line from II to III, the state with $\mu > 0$ goes from a stable state to a metastable state. In region IV, the energy reaches its absolute minimum only at the value of $\mu = 0$, the massive ground state is unstable. Finally, we would like to mention that the transition point for $\Lambda \sim \infty$ is $\beta^2 = 4\pi$; however, the corresponding result by Coleman is $\beta^2 = 8\pi$ for the ordinary sine–Gordon model. One would speculate if the transition point would interpolate between these two exact values when the delta function in the interaction term is replaced by some type of smooth distribution.

It is also interesting to connect the phase diagram to the low-temperature behaviour of the Kondo model and further to the quantum tunnelling system. It is well known that the Kondo model with anisotropic exchange can be approximately mapped to a scalar field model via the bosonization technique [8, 1]. The relations between the parameters are $\beta = \sqrt{4\pi}(1 - \rho J_{\|}/2)$ and $\alpha/\beta^2 = J_{\perp}/4\pi$. Thus the ferromagnetic exchange, $J_{\|} < 0$ or $\beta^2 > 4\pi$, corresponds to the massless regime, where the $\cos\beta\phi(0)$ is irrelevant and the weak coupling picture holds. On the other hand, the antiferromagnetic exchange $J_{\|} > 0$ corresponds to the massive regime, the ground state is strongly correlated and the scaling transformation flows to infinity. In the frame of the Gaussian wave-functional approach, the renormalized mass is given by equation (16).

Acknowledgments

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